Dimensions, Tangent Spaces and Smooth Affine Varieties

Ryan, Lok-Wing Pang

Department of Mathematics
Hong Kong University of Science and Technology

Introduction to Algebraic Geometry, 2015
Outline

1. Dimension of an Affine Variety
2. Tangent Spaces
3. Smooth Varieties: Motivations
Throughout the presentation, we assume the underlying field $k$ to be algebraically closed.
Dimension of an Affine Variety

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- Surprisingly, it turns out that the right definition is purely topological: it depends only on the topological space.
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- The idea to define dimension in algebraic topology using Zariski topology is the following:
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- Surprisingly, it turns out that the right definition is purely topological: it depends only on the topological space.

Definition (Dimension)

Let $X$ be a topological space. Then the dimension $\dim X$ of $X$ is defined to be the supremum of all integers $n$ such that there exists a chain of irreducible closed subsets of $X$:

$$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$$

contained in $X$. In particular, if $X$ is an affine variety, then we define the dimension of the variety to be its dimension as a topological space.
Example

Recall from last time we proved that the only proper subvarieties of $\mathbb{A}^1_k$ are the whole space and single points, hence $\dim \mathbb{A}^1_k = 1$. 

This definition has the advantage of being short and intuitive, but it has the disadvantage that it is very hard to apply in actual computations. To this end, we have an alternative and equivalent definition:

Theorem

Let $X$ be an affine variety. Then $\dim X = \text{tr.deg} \left( \mathbb{K}(X) / \mathbb{K} \right)$, the transcendence degree of $\mathbb{K}(X)$ over $\mathbb{K}$, where $\mathbb{K}(X)$ is the field of rational functions of $X$. 

Proving the equivalence requires a lot of commutative algebra.

Example

The function field $\mathbb{K}(\mathbb{A}^n_k) = \mathbb{K}(x_1, \ldots, x_n)$. Hence $\dim \mathbb{A}^n_k = n$. 

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Recall that we have a 1-1 correspondence between affine varieties $V$ and prime ideals in $k[x_1, \cdots, x_n]$: 
\[ \{ V \subseteq \mathbb{A}^n_k \} \longleftrightarrow \text{Spec } k[x_1, \cdots, x_n]. \]
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We also have:

**Theorem (Maximal Ideal - Point Correspondence)**

Let $X$ be an affine variety with coordinate ring $R = k[x_1, \cdots, x_n]/I(X)$. Then there is a bijective correspondence between the points $p \in X$ and the maximal ideals $m \subseteq R$. 
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Proof.

A maximal (hence prime) ideal $m$ of $k[x_1, \cdots, x_n]$ corresponds to a minimal affine variety of $\mathbb{A}^n_k$ (because the correspondence is inclusion reversing), which must be a point $p$. The result follows from the 4th isomorphism theorem.
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- It implies that every maximal ideal of $k[x_1, \cdots, x_n]$ is of the form $m = (x_1 - a_1, \cdots, x_n - a_n)$ for some $a_i \in k$. 

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In calculus, we define the tangent plane to a surface by taking the span of tangent vectors to arcs lying on it. We want to translate this into algebraic geometry.

**Definition (Derivations)**

Let $A \rightarrow B$ be a homomorphism of commutative algebras, and $M$ a $B$-module. We define the derivations $\text{Der}_A(B, M) = \{ D : B \rightarrow M | D \text{satisfies (1) and (2)} \}$

1. $D(b_1b_2) = D(b_1)b_2 + b_1D(b_2) \forall b_1, b_2 \in B$.
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If $A = k$, $B = k[x_1, \cdots, x_n]$, then $M = k$ is a $B$-module. Define the map $\partial/\partial x_i : B \to k$ by $x_j \mapsto \delta_{ij}$. We have

$$\text{Der}_k(B, k) = \left\{ D = \sum_i \lambda_i \frac{\partial}{\partial x_i} : \lambda_i \in k \right\}.$$
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Note that we have $k \cong \mathcal{O}_{X, p}/m_p$. This is because the map $k[X] \to k$ given by evaluation at $p$ has, by definition, kernel $m_p$, and so does the map $\mathcal{O}_{X, p} \cong k[X]_{m_p} \to k$. 
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Definition (Tangent Space)

Let $X$ be an affine variety. The Zariski Tangent Space of $X$ at $p$ is defined as $T_pX = \text{Der}_k(\mathcal{O}_{X,p}, k)$. 
Theorem

Let $X$ be an affine variety and $p \in X$, then $T_pX \cong (m_p/m_p^2)^*$, where $V^*$ is the dual vector space of $V$.

Proof.
Consider the map $T_pX \to \text{Hom}(m_p/m_p^2, k)$ defined by $D \to D|_{m_p}$. $D$ vanishes on $m_p^2$ because $D(fg) = fD(g) + D(f)g \equiv 0 \pmod{m_p}$ for $f, g \in m_p$. As a $k$-vector space, $\mathcal{O}_{X,p} \cong k \oplus m_p$, we see that $D$ is determined by $m_p$. Hence $D$ induces a linear map $D: m_p/m_p^2 \to k$ and $T_pX \to \text{Hom}(m_p/m_p^2, k)$ is injective. On the other hand, let $C = m_p \setminus m_p^2$ be the complement of $m_p^2$ so that $\mathcal{O}_{X,p} \cong k \oplus C \oplus m_p^2$. If $T: C \to k$ is linear, then it is clear that the extension of $T$ to a linear map $D$ on $\mathcal{O}_{X,p}$ by putting $T|_{k \oplus m_p^2} = 0$ is a derivation in $p$. 

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**Definition (Hypersurfaces)**

A *hypersurface* in $\mathbb{A}^n_k$ is a subvariety defined by a single equation $f(x_1, \ldots, x_n) = 0$. 

**Definition (Singularities of a Hypersurface)**

Let $X$ be a hypersurface $f(x_1, \ldots, x_n) = 0$ in $\mathbb{A}^n_k$. A point $p \in X$ is a singularity of $X$ if $\frac{\partial f}{\partial x_i}(P) = 0$ for all $i = 1, \ldots, n$. The set of singularities forms a subvariety of $X$, defined by $f = 0$ together with the equations $\frac{\partial f}{\partial x_i} = 0$ for all $i = 1, \ldots, n$.

**Definition (Smoothness)**

A hypersurface $X$ in $\mathbb{A}^n_k$ is called smooth (or nonsingular) if there are no singularities in $X$. 

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Motivating Example

Example (Elliptic Curve)

Let $E$ be the elliptic curve $y^2 = x^3 + 1$ in $\mathbb{A}^2_{\mathbb{Q}}$, then the singular locus is defined by the equations $y^2 - x^3 - 1 = -3x^2 = 2y = 0$, which have no common solution in $\overline{\mathbb{Q}}$, so the curve is smooth.
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Example (Nodes in a Singular Curve)

Let $X$ be the "nodal cubic" $y^2 = x^3 + x^2$, then the singular locus is defined by the equations $y^2 - x^3 - x^2 = -3x^2 - 2x = 2y = 0$, which have a unique solution $(0, 0)$. Hence $X$ is singular. It has two "branches" crossing at $(0, 0)$. Such a singularity is called a node.
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Example (Cusps in a Singular Curve)
Let $Y$ be the curve $y^2 = x^3$, then $Y$ is singular with a unique singularity $(0, 0)$. It has no tangent at $(0, 0)$. Such a singularity is called a cusp.
More generally:

**Definition (Smoothness)**

An affine variety \( X = V(f_1, \cdots, f_m) \) is **smooth (or nonsingular)** at \( p \) if the \( m \times n \) Jacobian matrix has rank \( n - \dim X \), i.e.

\[
\text{rk} \left( \frac{\partial f_i}{\partial x_j} (p) \right)_{1 \leq i \leq m, 1 \leq j \leq n} = n - \dim X.
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\( X \) is nonsingular (or smooth) if \( X \) is nonsingular at every point.
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- When \( X \) is generated by a single non-constant polynomial, then the definition reduces to the case of hypersurface.
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- When \( X \) is generated by a single non-constant polynomial, then the definition reduces to the case of hypersurface.
- If \( X = V(f) \) is a hypersurface, then \( p \in X \) is a singularity iff \( \frac{\partial f}{\partial x_i} (p) = 0 \) for all \( i = 1, \cdots, n \) and \( f(P) = 0 \), this gives \( n + 1 \) equations in \( n \) variables. Thus for a randomly chosen hypersurface \( X \), we would expect \( X \) to be nonsingular.
Equivalent Definitions of Smoothness

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**Theorem (Zariski, 1947)**

Let $X \subseteq \mathbb{A}^n_k$ be an affine variety and $p \in X$. Then $X$ is nonsingular at $P$ iff

$$\dim k[X]/m_p(m_p/m_p^2) = \dim X.$$
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**Theorem (Zariski, 1947)**

Let $X \subseteq \mathbb{A}_k^n$ be an affine variety and $p \in X$. Then $X$ is nonsingular at $P$ iff

$$\dim_{k[X]/m_P}(m_P/m_P^2) = \dim X.$$ 

- This theorem is important because it implies that nonsingularity is intrinsic, i.e. it could be described in terms of the local rings, rather than a set of generators.
Proof.

First we can identify \( k[X]/\mathfrak{m}_p \) with \( k \) (why?).
Equivalent Definitions of Smoothness

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\[
\varphi : k[x_1, \cdots, x_n] \to k^n \text{ by } \varphi(f) = (\frac{\partial f}{\partial x_1}(p), \cdots, \frac{\partial f}{\partial x_n}(p)).
\]

Then it is clear that \( \varphi(x_i - a_i), i = 1, \cdots, n \) form a basis of \( k^n \) and \( \varphi(a_p) = 0 \). Hence \( \varphi \) induces an isomorphism \( \varphi : a_p/\mathfrak{a}_p \to k^n \).

Now write \( X = V(\mathfrak{b}) \) and let \( f_1, \cdots, f_t \) be a set of generators of \( \mathfrak{b} \subset k[X] \) (this can be done because \( k[X] \) is Noetherian).

Then \( \text{rk}(\frac{\partial f_i}{\partial x_j}(p)) = \dim \varphi(\mathfrak{b}) \).

Using the isomorphism \( \varphi \), this is the same as \( \dim \mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}_p) = \dim(\mathfrak{b} + \mathfrak{a}_p)/\mathfrak{a}_p \).

Now, \( \mathcal{O}_{X,p} \sim = k[X]_{a_p} \).

Hence if \( m \) is the maximal ideal of \( \mathcal{O}_{X,p} \), we have \( m/m^2 \sim = \mathfrak{a}_p/(\mathfrak{b} + \mathfrak{a}_p) \).

Counting dimensions of vector spaces, we have \( \dim m/m^2 + \text{rk}(\frac{\partial f_i}{\partial x_j}(p)) = n \).

The result follows immediately.
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Hence $\varphi$ induces an isomorphism $\overline{\varphi} : a_p/a_p^2 \to k^n$.
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Then it is clear that $\varphi(x_i - a_i), i = 1, \cdots, n$ form a basis of $k^n$ and $\varphi(a_p^2) = 0$. Hence $\varphi$ induces an isomorphism $\overline{\varphi} : a_p/\mathfrak{a}_p^2 \to k^n$. Now write $X = V(\mathfrak{b})$ and let $f_1, \cdots, f_t$ be a set of generators of $\mathfrak{b} \subseteq k[x_1, \cdots, x_n]$ (this can be done because $k[x_1, \cdots, x_n]$ is Noetherian).
Proof.

First we can identify \( k[X]/m_p \) with \( k \) (why?). Let \( p = (a_1, \cdots, a_n) \in \mathbb{A}^n_k \) and let \( a_p = (x_1 - a_1, \cdots, x_n - a_n) \). We define a map by
\[
\varphi : k[x_1, \cdots, x_n] \rightarrow k^n \text{ by } \varphi(f) = (\partial f/\partial x_1(p), \cdots, \partial f/\partial x_n(p)).
\]
Then it is clear that \( \varphi(x_i - a_i), i = 1, \cdots, n \) form a basis of \( k^n \) and \( \varphi(a_p^2) = 0 \).
Hence \( \varphi \) induces an isomorphism \( \overline{\varphi} : a_p/a_p^2 \rightarrow k^n \). Now write \( X = V(\mathfrak{b}) \) and let \( f_1, \cdots, f_t \) be a set of generators of \( \mathfrak{b} \subseteq k[x_1, \cdots, x_n] \) (this can be done because \( k[x_1, \cdots, x_n] \) is Noetherian). Then
\[
\text{rk}(\partial f_i/\partial x_j(p)) = \dim \varphi(\mathfrak{b}).
\]
Equivalent Definitions of Smoothness

Proof.

First we can identify $k[X]/\mathfrak{m}_p$ with $k$ (why?). Let $p = (a_1, \cdots, a_n) \in \mathbb{A}^n_k$ and let $\mathfrak{a}_p = (x_1 - a_1, \cdots, x_n - a_n)$. We define a map by

$$\varphi : k[x_1, \cdots, x_n] \to k^n \text{ by } \varphi(f) = (\partial f / \partial x_1(p), \cdots, \partial f / \partial x_n(p)).$$

Then it is clear that $\varphi(x_i - a_i), i = 1, \cdots, n$ form a basis of $k^n$ and $\varphi(\mathfrak{a}_p^2) = 0$. Hence $\varphi$ induces an isomorphism $\overline{\varphi} : \mathfrak{a}_p / \mathfrak{a}_p^2 \to k^n$. Now write $X = V(\mathfrak{b})$ and let $f_1, \cdots, f_t$ be a set of generators of $\mathfrak{b} \subseteq k[x_1, \cdots, x_n]$ (this can be done because $k[x_1, \cdots, x_n]$ is Noetherian). Then

$$rk(\partial f_i / \partial x_j(p)) = \dim \varphi(\mathfrak{b}).$$

Using the isomorphism $\overline{\varphi}$, this is the same as

$$\dim \mathfrak{b} / (\mathfrak{b} \cap \mathfrak{a}_p^2) = \dim (\mathfrak{b} + \mathfrak{a}_p^2) / \mathfrak{a}_p^2.$$
Proof.

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\]
Then it is clear that \( \varphi(x_i - a_i), i = 1, \cdots, n \) form a basis of \( k^n \) and \( \varphi(a^2_p) = 0 \).

Hence \( \varphi \) induces an isomorphism \( \overline{\varphi} : a_p/a^2_p \to k^n \). Now write \( X = V(b) \) and let \( f_1, \cdots, f_t \) be a set of generators of \( b \subseteq k[x_1, \cdots, x_n] \) (this can be done because \( k[x_1, \cdots, x_n] \) is Noetherian). Then
\[
\text{rk}(\partial f_i/\partial x_j(p)) = \dim \varphi(b).\quad \text{Using the isomorphism } \overline{\varphi}, \text{ this is the same as}
\]
\[
\dim b/(b \cap a^2_p) = \dim (b + a^2_p)/a^2_p. \quad \text{Now, } \mathcal{O}_{X,p} \cong k[X]_{a_p}.
\]
Proof.

First we can identify $k[X]/\mathfrak{m}_p$ with $k$ (why?). Let $p = (a_1, \cdots, a_n) \in \mathbb{A}^n_k$ and let $a_p = (x_1 - a_1, \cdots, x_n - a_n)$. We define a map by

$$\varphi : k[x_1, \cdots, x_n] \to k^n \text{ by } \varphi(f) = \left( \frac{\partial f}{\partial x_1}(p), \cdots, \frac{\partial f}{\partial x_n}(p) \right).$$

Then it is clear that $\varphi(x_i - a_i), i = 1, \cdots, n$ form a basis of $k^n$ and $\varphi(a_p^2) = 0$. Hence $\varphi$ induces an isomorphism $\overline{\varphi} : a_p/a_p^2 \to k^n$. Now write $X = V(\mathfrak{b})$ and let $f_1, \cdots, f_t$ be a set of generators of $\mathfrak{b} \subseteq k[x_1, \cdots, x_n]$ (this can be done because $k[x_1, \cdots, x_n]$ is Noetherian). Then

$$\text{rk} \left( \frac{\partial f_i}{\partial x_j}(p) \right) = \dim \varphi(\mathfrak{b}).$$

Using the isomorphism $\overline{\varphi}$, this is the same as

$$\dim \mathfrak{b} / (\mathfrak{b} \cap a_p^2) = \dim (\mathfrak{b} + a_p^2) / a_p^2.$$

Now, $\mathcal{O}_{X,p} \cong k[X]_{a_p}$. Hence if $m$ is the maximal ideal of $\mathcal{O}_{X,p}$, we have $m/m^2 \cong a_p/(\mathfrak{b} + a_p^2)$. 

Equivalent Definitions of Smoothness

Proof.

First we can identify $k[X]/\mathfrak{m}_p$ with $k$ (why?). Let $p = (a_1, \cdots, a_n) \in \mathbb{A}_k^n$ and let $\mathfrak{a}_p = (x_1 - a_1, \cdots, x_n - a_n)$. We define a map by

$\varphi : k[x_1, \cdots, x_n] \to k^n$ by $\varphi(f) = (\partial f / \partial x_1(p), \cdots, \partial f / \partial x_n(p))$. Then it is clear that $\varphi(x_i - a_i), i = 1, \cdots, n$ form a basis of $k^n$ and $\varphi(\mathfrak{a}_p^2) = 0$. Hence $\varphi$ induces an isomorphism $\overline{\varphi} : \mathfrak{a}_p / \mathfrak{a}_p^2 \to k^n$. Now write $X = V(\mathfrak{b})$ and let $f_1, \cdots, f_t$ be a set of generators of $\mathfrak{b} \subseteq k[x_1, \cdots, x_n]$ (this can be done because $k[x_1, \cdots, x_n]$ is Noetherian). Then

$rk(\partial f_i / \partial x_j(p)) = \dim \varphi(\mathfrak{b})$. Using the isomorphism $\overline{\varphi}$, this is the same as

$\dim \mathfrak{b} / (\mathfrak{b} \cap \mathfrak{a}_p^2) = \dim (\mathfrak{b} + \mathfrak{a}_p^2) / \mathfrak{a}_p^2$. Now, $\mathcal{O}_{X,p} \cong k[X]_{\mathfrak{a}_p}$. Hence if $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$, we have $\mathfrak{m} / \mathfrak{m}^2 \cong \mathfrak{a}_p / (\mathfrak{b} + \mathfrak{a}_p^2)$. Counting dimensions of vector spaces, we have $\dim \mathfrak{m} / \mathfrak{m}^2 + rk(\partial f_i / \partial x_j(p)) = n$. The result follows immediately.
Equivalent Definitions of Smoothness

Proof.

First we can identify $k[X]/m_p$ with $k$ (why?). Let $p = (a_1, \cdots, a_n) \in \mathbb{A}_k^n$ and let $a_p = (x_1 - a_1, \cdots, x_n - a_n)$. We define a map by

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Then it is clear that $\varphi(x_i - a_i), i = 1, \cdots, n$ form a basis of $k^n$ and $\varphi(a_p^2) = 0$. Hence $\varphi$ induces an isomorphism $\overline{\varphi} : a_p/a_p^2 \rightarrow k^n$. Now write $X = V(b)$ and let $f_1, \cdots, f_t$ be a set of generators of $b \subseteq k[x_1, \cdots, x_n]$ (this can be done because $k[x_1, \cdots, x_n]$ is Noetherian). Then

$$rk(\partial f_i / \partial x_j(p)) = \dim \varphi(b).$$

Using the isomorphism $\overline{\varphi}$, this is the same as

$$\dim b / (b \cap a_p^2) = \dim (b + a_p^2) / a_p^2.$$  

Now, $\mathcal{O}_{X,p} \cong k[X]_{a_p}$. Hence if $m$ is the maximal ideal of $\mathcal{O}_{X,p}$, we have $m/m^2 \cong a_p / (b + a_p^2)$. Counting dimensions of vector spaces, we have $\dim m/m^2 + rk(\partial f_i / \partial x_j(p)) = n$. The result follows immediately.
In fact, an even stronger result is true:
In fact, an even stronger result is true:

**Theorem**

Let \( X \subseteq \mathbb{A}^n_k \) be an affine variety with coordinate ring \( k[X] \cong \mathcal{O}(X) \), then \( X \) is nonsingular at \( p \) iff the localization \( k[X]_{m_p} \cong \mathcal{O}_{X,p} \) of \( k[X] \) at \( m_p \) is a UFD.

- Proving the equivalence requires a lot of commutative algebra.
Equivalent Definitions of Smoothness

To summarize:

\[ \begin{align*}
(1) & \quad \text{\(X\) is smooth (or nonsingular) at \(p\),} \\
(2) & \quad \text{the \(m \times n\) Jacobian matrix has rank \(n - \dim X\), i.e.} \\
& \quad \text{\(\text{rk} \left( \frac{\partial f_i}{\partial x_j} \right) \bigg|_{p} \right) \leq i \leq m, \quad 1 \leq j \leq n = n - \dim X,\)} \\
(3) & \quad \dim \left( k[X] / \mathfrak{m}_p \mathfrak{m}_p^2 \right) = \dim X, \\
(4) & \quad \text{The localization} \quad k[X]_{\mathfrak{m}_p} \cong O_{X, p}.
\end{align*} \]

Food for thought: Verify that the localization of \(k[\mathbb{A}^2] / (y^2 - x^3)\) and \(k[\mathbb{A}^2] / (y^2 - x^3 - x^2)\) at \(p = (0,0)\) is not a UFD.
Equivalent Definitions of Smoothness

To summarize:

**Theorem**

Let $X = V(f_1, \cdots, f_m) \subseteq \mathbb{A}_k^n$ be an affine variety and let $p \in X$. Then the following are equivalent:

1. $X$ is smooth (or nonsingular) at $p$,
2. the $m \times n$ Jacobian matrix has rank $n - \dim X$, i.e.
   
   $$ \text{rk} \left( \frac{\partial f_i}{\partial x_j} (p) \right)_{1 \leq i \leq m, 1 \leq j \leq n} = n - \dim X, $$

3. $\dim_k [X]/m_p (m_p/m_p^2) = \dim X,$

4. The localization $k[X]_{m_p} \cong \mathcal{O}_{X,p}$ of $k[X]$ at $m_p$ is a UFD.
Equivalent Definitions of Smoothness

To summarize:

**Theorem**

Let $X = V(f_1, \cdots, f_m) \subseteq \mathbb{A}_k^n$ be an affine variety and let $p \in X$. Then the following are equivalent:

1. $X$ is smooth (or nonsingular) at $p$,
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   \[
   \text{rk} \left( \frac{\partial f_i}{\partial x_j}(p) \right)_{1 \leq i \leq m, 1 \leq j \leq n} = n - \dim X,
   \]
3. $\dim_{k[X]}(m_p/m_p^2) = \dim X$,
4. The localization $k[X]_{m_p} \cong \mathcal{O}_{X,p}$ of $k[X]$ at $m_p$ is a UFD.

- **Food for thought:** Verify that the localization of $k[x, y]/(y^2 - x^3)$ and $k[x, y]/(y^2 - x^3 - x^2)$ at $p = (0, 0)$ is not a UFD.
References


Thanks

Happy Easter everyone!